SIMPLY TRANSITIVE PRIMITIVE GROUPS*

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1. The transitive constituents of the subgroup that leaves fixed one letter of a transitive group occur in pairs of equal degree. Transitive constituents on one letter are to be taken into account in the above statement. The two members of a pair sometimes coincide. This important property of transitive groups was proved by means of a certain quadratic invariant by Burnside in 1900.† It can be more easily demonstrated as follows:

Let G be a non-regular transitive group and let the subgroup G_1 of order g_1 that fixes one letter a of G have the transitive constituents B on the letters $b, b_1, \cdots, b_{r-1}$; C on the letters $c, c_1, \cdots, c_{s-1}$; $\cdots$. Consider a permutation $S=(cab \cdots) \cdots$. Every one of the g_1 permutations G_1S replaces a by b; and because $c, c_1, \cdots$ are the letters of a transitive constituent of G_1, every permutation G_1S is of the form $(c'ab \cdots) \cdots$, where c' is some one of the letters $c, c_1, \cdots$. Similarly every permutation SG_1 is of the form $(cab' \cdots) \cdots$. Then the array G_1SG_1 includes every permutation of G in which a is preceded by one of the s c''s or is folowed by one of the r b''s. Now the number of distinct permutations in the array G_1SG_1 is g_1^2 divided by the number of permutations common to G_1 and SG_1S^{-1}, or common to $S^{-1}G_1S$ and G_1,‡ that is, by the number of permutations of G_1 that fix c or b. These numbers are g_1/s and g_1/r. Therefore $r=s$. If, as often happens, every permutation $(ab \cdots) \cdots$ is of the form $(ab) \cdots$ or $(b_1ab \cdots) \cdots$, the transitive constituent B is paired with itself. Since the product

$$(a)(bb_1 \cdots) \cdots (b_1ab \cdots) \cdots = (ab) \cdots,$$

G, whenever a transitive constituent of G_1 is paired with itself, is of even order. A properly chosen odd power of this product is a permutation $(ab) \cdots$ of order a power of 2.

2. As in §1, G is a transitive group of order g and G_1 is a subgroup that fixes one of the n letters of G. Let H be a subgroup of G_1 of degree $n-m$ $(0<m<n)$ and let I be the largest subgroup of G in which H is invariant.§ If H is one of r conjugate subgroups in G_1, the largest subgroup of G_1 in which

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† Burnside, Proceedings of the London Mathematical Society, vol. 33 (1901), p. 162.

‡ Miller, these Transactions, vol. 12 (1911), p. 326.

§ Manning, these Transactions, vol. 19 (1918), p. 129.

H is invariant is of order g/nr and the order of I is gm/ns, where s is the total number of conjugates of H under G found in G_1. Then I has a transitive constituent of degree mr/s in letters fixed by H. The letters of this constituent are $a, a_1, \cdots$ and a is the letter fixed by G_1. Every permutation $(aa_1 \cdots) \cdots$ of G transforms H into one of its r conjugates $H, H', \cdots$ under G_1. A second set of r_1 conjugate subgroups of G_1 is $H_1, H_1', \cdots$. Each of the g_1 permutations $(ab \cdots) \cdots$ of G transforms one of the r_1 subgroups $H_1, H_1', \cdots$ into H, so that in G_2, the subgroup of G that fixes b, H is one of r_1 conjugates and I has a transitive constituent of degree mr_1/s on the letters $b, b_1, \cdots$. The two sets of letters $a, a_1, \cdots$ and $b, b_1, \cdots$ do not coincide. If there is a third conjugate set of r_2 members $H_2, H_2', \cdots$ in G_1, there is a third transitive constituent $c, c_1, \cdots$ of degree mr_2/s in I, and so on. It is this correspondence between the conjugate sets $H, H', \cdots; H_1, H_1', \cdots; \cdots$ of G_1 and the transitive constituents $a, a_1, \cdots; b, b_1, \cdots; \cdots$ of I in the letters fixed by H, which is to be borne in mind in the developments of the following sections.

3. Examples of transitive groups in illustration of the preceding theory may be helpful.

$$\begin{aligned}
\overset{5}{G}_{10} &= \{(bb_1)(cc_1),\ (ab)(b_1c)\}.\\
\overset{7}{G}_{21} &= \{(bb_1b_2)(cc_1c_2),\ (abc)(b_1c_2b_2)\}.\\
\overset{9}{G}_{72} &= \{(bb_1)(b_2b_3)(cc_1),\ (bb_2)(cc_2)(c_1c_3),\ (ab)(b_1c_2)(b_3c_1)\}.\\
\overset{10}{G}_{120} &= \{(bb_1)(cc_4)(c_1c_5),\ (bb_2)(c_2c_4)(c_3c_5),\ (cc_1)(c_2c_3)(c_4c_5),\\
&\qquad (ab)(b_1c_5)(b_2c_4)(c_1c_2)\}.
\end{aligned}$$

These four groups are primitive. In the one of degree 10 the subgroup that leaves a fixed is the following:

$$\begin{array}{ll}
1, & (cc_1)(c_2c_3)(c_4c_5),\\
(bb_1b_2)(cc_3c_4c_1c_2c_5), & (bb_1b_2)(cc_2c_4)(c_1c_3c_5),\\
(bb_2b_1)(cc_5c_2c_1c_4c_3), & (bb_2b_1)(cc_4c_2)(c_1c_5c_3),\\
(bb_1)(cc_4)(c_1c_5), & (bb_1)(cc_5)(c_1c_4)(c_2c_3),\\
(bb_2)(c_2c_4)(c_3c_5), & (bb_2)(cc_1)(c_2c_5)(c_3c_4),\\
(b_1b_2)(cc_2)(c_1c_3), & (b_1b_2)(cc_3)(c_1c_2)(c_4c_5).
\end{array}$$

4. *In all that follows G is a simply transitive primitive group.* G_1 is intransitive of degree $n-1$ and is a maximal subgroup of G. Let H be invariant in G_1 and of degree $<n-1$. Then I and G_1 coincide and I may be said to have one transitive constituent on the one letter a fixed by G_1. Hence the factor m/s of §2 is unity. Since H is invariant in G_1, it must fix all the letters of

one or more transitive constituents of G_1. Because G_1 does not fix two letters of G, of all the conjugates of H under G, only H is invariant in G_1. Therefore H, like the letter a, is characteristic of G_1. It is not, however, to be inferred from this statement that H is a "characteristic" subgroup of G_1 in the strong sense of being invariant in the holomorph of G_1. What is meant is that the n conjugate subgroups $H, \cdots$ are in one-to-one correspondence to the n subgroups $G_1, \cdots$ and therefore are in one-to-one correspondence to the n letters $a, \cdots$ of G. In G_1 there are exactly $m-1$ non-invariant subgroups $H_1, \cdots$, and "they are transformed by G_1 in the same manner as the letters of one of G_1's constituent groups of degree $m-1$."* The constituent group may be transitive or intransitive. This well known conclusion leaves open the question as to whether or not this constituent of G_1 according to which the $m-1$ subgroups $H_1, \cdots$ are permuted contains letters displaced by H.† This is an unsolved problem of fundamental importance.

If the transitive constituent B (on letters fixed by H) is paired with itself in the sense of §1, the permutation $S=(ab)\cdots$, known to exist in G, which transforms H_1 into H, has an inverse $S^{-1}=(ab)\cdots$ which transforms H into H_1 and which transforms some member H_1' of the conjugate set $H_1, H_1', \cdots$ of G_1 into H (§2). Now H_1 is the invariant subgroup of the subgroup G_2 that fixes b and in which H is included. The r_1 conjugate subgroups $H_1, H_1', \cdots$ are therefore permuted according to the permutations of the transitive constituent B of G_1. Conversely, if in the permutation $S=(ab)\cdots$ of §2 which transforms H_1 into H, the letter b is one of the transitive set according to which $H_1, H_1', \cdots$ are permuted by G_1, B is paired with itself.

If two transitive constituents B and C, both in letters not displaced by H, are paired, there is in G a permutation $S=(cab\cdots)\cdots$ such that

$$S^{-1}H_1S = H, \qquad SH_2S^{-1} = H;$$

and hence

$$SHS^{-1} = H_1, \qquad S^{-1}HS = H_2.$$

Then $H_1, H_1', \cdots$ are permuted according to the transitive constituent C, and $H_2, H_2', \cdots$ are permuted according to the transitive constituent B. Here also the converse is true.

5. Theorem I. *If all the transitive constituents of H are of the same degree, or if no two (not of maximum degree in H) belong to the same transitive constituent of G_1, every subgroup of G_1 similar to H is transformed into itself by H.*

* Miller, Proceedings of the London Mathematical Society, vol. 28 (1897), p. 535.

† Rietz, American Journal of Mathematics, vol. 28 (1904), p. 10, line 23.

The above conclusion is equivalent to the statement that the constituent of G_1 according to which the $m-1$ subgroups $H_1, H_1', \cdots$ are permuted displaces no letter of H. For if $H_1, H_1', \cdots$ are permuted according to the constituent B of G_1 on the r_1 letters $b, b_1, \cdots$ fixed by H, the subgroup of G_1 that fixes b, say, is the largest subgroup of G_1 in which H_1 is invariant and includes H. Conversely, if each of the r_1 subgroups $H_1, H_1', \cdots$ is transformed into itself by every permutation of H, the transitive constituent of degree r_1 of G_1 according to which they are permuted displaces no letter of H.

The letter of G fixed by G_1 is a. Let $b, b_1, \cdots$ be certain letters fixed by H but permuted transitively by G_1. Let $\alpha, \alpha_1, \cdots, \beta, \beta_1, \cdots, \cdots$ be the letters of some transitive constituent of G_1 and such that $\alpha, \alpha_1, \cdots$ is one transitive constituent of H, $\beta, \beta_1, \cdots$ is another, and so on. Of course if H displaces one letter of a transitive constituent of G_1 it displaces every letter of that constituent.

Now transform G_1 into G_2 by means of a permutation $(ab \cdots) \cdots$ of the primitive group G. At the same time H, an invariant subgroup of G_1, is transformed into an invariant subgroup of G_2. Call the latter subgroup H_b. We wish to show that H_b is necessarily a subgroup of G_1. Suppose it is not a subgroup of G_1. Then since a primitive group is generated by a subgroup leaving one letter fixed and any permutation of the group not in that subgroup, $\{G_1, H_b\} = G$. But if H_b fails to connect transitively letters of H and letters fixed by H, $\{G_1, H_b\}$ is intransitive. Hence, if H_b is not a subgroup of G_1, at least one of its permutations unites letters of H and letters fixed by H. Let us now impose upon the transitive constituent $\alpha, \alpha_1, \cdots$ of H the condition that no transitive constituent of H is of higher degree. The set $\beta, \beta_1, \cdots$, being in the same transitive constituent of G_1, will have exactly the same number of letters as the set $\alpha, \alpha_1, \cdots$. If α and x (let x be one of the m letters $a, b, b_1, \cdots$ fixed by H) are in the same transitive constituent of H_b, so also are all the other letters $\alpha_1, \alpha_2, \cdots$ of that transitive constituent of H. For since H fixes b, it is a subgroup of G_2, and in consequence every permutation of H transforms H_b into itself. Then H_b has a transitive constituent $x, \alpha, \alpha_1, \cdots$ of higher degree than any transitive constituent of H, to which H_b is conjugate under G; —an absurd result. Similarly the constituent $\beta, \beta_1, \cdots$ of H_b displaces no letter fixed by H. If H_b fixes all the letters $\alpha, \beta, \cdots$ of one transitive constituent of G_1, the group $\{G_1, H_b\}$ is intransitive. Then H_b connects the letters $\alpha, \beta, \cdots$ only with letters of H. Thus if all the transitive constituents of H are of the same degree the theorem is proved.

The letters $\lambda, \lambda_1, \cdots$ of a transitive constituent of H of lower degree are by hypothesis the letters of a transitive constituent of G_1. The transitive constituent $x, \lambda, \lambda_1, \cdots$ of H_b contains all the letters of the transitive constituent $\lambda, \lambda_1, \cdots$ of G_1. Then the transitive constituents $\alpha, \cdots$ and $x, \lambda, \cdots$ of H_b are not united by G_1.

COROLLARY I. *If G_1 has only two transitive constituents and contains an invariant subgroup H of degree $<n-1$, every subgroup of G_1 similar to H is transformed into itself by H, and G is of even order.*

In this case H is an invariant intransitive subgroup of an imprimitive group and all its transitive constituents are of the same degree. It is of even order because each transitive constituent of G_1 is paired with itself.

It was proved by Rietz* that if G is of odd order and if G_1 has only two transitive constituents, G_1 is a simple isomorphism between its two constituents.

COROLLARY II. *If G is of even order and G_1 has only two transitive constituents, each transitive constituent of G_1 is paired with itself.*

For G certainly contains a permutation $(ab) \cdots$ of order 2 which pairs one of the transitive constituents (B) with itself.

COROLLARY III. *If G_1 has three and only three transitive constituents and contains an invariant subgroup H of degree $<n-1$, every subgroup of G_1, similar to H, is transformed into itself by H.*

This is true except perhaps when H has transitive constituents $\alpha, \alpha_1, \cdots, \beta, \beta_1, \cdots, \cdots$ of degree t; and transitive constituents $\lambda, \lambda_1, \cdots, \mu, \mu_1, \cdots, \cdots$ of degree v $(<t)$. The letters fixed by H are $b, b_1, \cdots$ of the transitive constituent B.

Assume as before that H_b displaces a, that is, H_b is not a subgroup of G_1. If the only letters in the transitive constituent of H_b with a are letters of B, G contains a permutation $(b'ab_1 \cdots) \cdots$, where b' is a letter of B, and the constituent B is paired with itself. This would prove the Corollary, so that H_b has a transitive constituent $a, \lambda, \cdots$ of degree t. Then G_1 is transformed into G_2 by a permutation $S=(\alpha ab \cdots) \cdots$ which pairs the constituents $b, b_1, \cdots$ and $\alpha, \alpha_1, \cdots$ of G_1. If the transitive constituent $a, \lambda, \cdots$ of H_b contains a letter of B, H_b contains a permutation $(ab_1 \cdots) \cdots$ in which a is preceded by a letter of B or by one of the letters $\lambda, \cdots, \mu, \cdots$. But every permutation of G which replaces a by a b belongs to the array G_1SG_1, in which only α's precede a. Now the transitive constituent $a, \lambda, \lambda_1, \cdots$ of H_b is of degree $t=kv+1$ if it contains letters of k transitive constituents of H. In one of the transitive constituents of H_b are found letters α and letters μ,

* Rietz, loc. cit., p. 11, Theorem 10.

for only thus can H_b unite these two transitive constituents of G_1. This transitive constituent of degree v in H_b cannot displace all the t letters $\alpha, \alpha_1, \cdots, \alpha_{t-1}$. But H_b, being transformed into itself by H, displaces all the $t+v$ letters $\alpha, \alpha_1, \cdots, \mu, \mu_1, \cdots$. H permutes q (say) transitive constituents $\alpha, \mu, \cdots$ of H_b. This means that the transitive constituent α of H has q systems of imprimitivity of t/q letters and that the transitive constituent μ of H has q systems of imprimitivity of v/q letters. That t and v should have a common factor q is inconsistent with the preceding result, $t=kv+1$. Therefore H_b is a subgroup of G_1.

6. THEOREM II. *Let G be a simply transitive primitive group in which each of the m subgroups $H, H_1, \cdots$ of G_1 is transformed into itself by every permutation of H, the invariant subgroup of G_1. If the degree of the group generated by the complete set of conjugates $H_1, H_1', \cdots$ of G_1 is less than $n-1$, the letters of G_1 left fixed by it are the letters of one or more of the transitive constituents whose letters are already fixed by H.*

The group K generated by the r_1 conjugates $H_1, H_1', \cdots$ is an invariant subgroup of G_1. By hypothesis K is of degree $<n-1$. There is therefore in G_1 a complete set of conjugate subgroups $K_1, K_1', \cdots$, similar to K, permuted according to a transitive constituent X of G_1. Now K_1 is conjugate to K under transformation by some permutation of G, so that K_1 is generated by some of the n subgroups $H, H_1, \cdots$. Because K_1 fixes a, its r_1 generating subgroups are subgroups of G_1. All the permutations of X except the identity actually permute two or more of the subgroups $K_1, K_1', \cdots$, conjugate under G_1. Then the identity is the only permutation of X that can occur in a constituent of H, because by hypothesis every permutation of H transforms each of the m subgroups $H, H_1, \cdots$ of G_1 into itself. Then X is one of the transitive constituents $B, C, \cdots$ of G_1 that displace no letter of H. What is true of $K_1, K_1', \cdots$ is true of all such conjugate sets of non-invariant subgroups of G_1 similar to K. Hence the only letters of G_1 fixed by K are letters already fixed by H.

7. THEOREM III. *If only one transitive constituent of G_1 is an imprimitive group (of order f), G_1 is of order f.*

Let B be the imprimitive constituent of G_1. Suppose that a subgroup H of G_1 corresponds to the identity of B. All the $n-m$ letters of v primitive constituents of G_1 are displaced by H. The $m-1$ other letters of G_1 are distributed among w transitive constituents $B, C, \cdots$. Since an invariant subgroup of a primitive group is transitive, no two transitive constituents of H belong to the same transitive constituent of G_1. Then by Theorem I, each of the $m-1$ non-invariant subgroups $H_1, H_1', \cdots$ similar to H of G_1 is transformed into itself by H. By Theorem II, the group K generated by the conjugate set $H_1, H_1', \cdots$ of G_1 displaces all the letters of H. Since K is not H or a subgroup of H, K also displaces the letters of the imprimitive constituent B. If one of the generators $H_1, H_1', \cdots$ of K fixes all the letters of a transitive constituent of G_1, K fixes all the letters of that constituent. Hence H_1 has $v+1$ or more transitive constituents. Under G, H is conjugate to H_1 and therefore also has $v+1$ or more transitive constituents. But H displaces the letters of v primitive constituents of G_1 and has exactly v transitive constituents. Hence it is impossible that the order of G_1 exceeds that of the imprimitive constituent B.

COROLLARY I. *If all the transitive constituents of G_1 are primitive groups, G_1 is a simple isomorphism between its transitive constituents.*

Each of the primitive constituents of G_1 may be put in turn in the place of the imprimitive constituent of Theorem III.

COROLLARY II. *If G_1 has an intransitive constituent of order f, and if all the other transitive constituents of G_1 are primitive groups, G_1 is of order f.*

8. It has been known since 1921 that if one of the transitive constituents of G_1 of maximum degree is doubly transitive, G_1 is a simple isomorphism between its transitive constituents. Moreover the transitive constituents are similar groups whose corresponding permutations are multiplied together. For this is an immediate consequence of the following theorem:*

THEOREM IV. *Let G_1 have a t-ply ($t \geqq 2$) transitive constituent of degree m. If G_1 has no transitive constituent whose degree is a divisor ($>m$) of $m(m-1)$, all the transitive constituents of G_1 are similar groups of order g/n.*

To guard against misunderstanding, we recall that two groups G and G' are *equivalent* when there exists a permutation by which one can be transformed into the other; and that two equivalent groups are *similar* when, if S_i and S_i' are corresponding permutations in some isomorphism of G to G', a permutation T exists such that $T^{-1}S_iT=S_i'$ $(i=1, 2, \cdots, g)$.† In the group G_{72}^9 of §3, the two octic constituents of G_1 are isomorphic and equivalent but are not similar. In G_{10}^5 and G_{21}^7 of §3, the two constituents of G_1 are similar.

A useful set of theorems having to do with simply transitive primitive groups was given by Dr. E. R. Bennett in 1912.‡ In particular, Corollary II to Theorem V, page 6, reads:

* Manning, *Primitive Groups*, 1921, p. 83.

† Manning, *Primitive Groups*, 1921, p. 39.

‡ E. R. Bennett, American Journal of Mathematics, vol. 34 (1912), p. 1.

"If the transitive constituent of degree m in G_1 is a t-times transitive group ($t \geqq 3$), G_1 always contains an imprimitive group of degree $m(m-1)$."

This result, in common with Dr. Bennett's Theorems I to VI, is subject to the following conditions upon the simply transitive primitive group G (of degree n) and its maximal subgroup G_1:

(1) The constituent M (of degree m) of G_1 is a non-regular transitive group.

(2) M is the only transitive constituent of G_1 whose degree divides m.

(3) Corresponding to the identity of M there is a subgroup H in G_1 of degree $n-m-1$.

This corollary raises interesting questions as to possible extensions of our Theorem IV, the proof of which is based merely upon the hypothesis that one constituent of G_1 is (at least) doubly transitive. Conditions (2) and (3) may be replaced by the weaker conditions of the following theorem:

THEOREM V. *Let G_1, the subgroup that leaves fixed one letter of the simply transitive primitive group G of degree n and order g, have a primitive constituent M of degree m, in which the subgroup M_1 that fixes one letter is primitive. Let M be paired with itself in G_1 and let the order of M be $<g/n$. Then G_1 contains an imprimitive constituent in which there is an invariant intransitive subgroup with m transitive constituents of $m-1$ letters each, permuted according to the permutations of the primitive group M.*

The letter of G fixed by G_1 is x, and the letters of M are $a, a_1, \cdots, a_{m-1}$. The subgroup of G that fixes both x and a is F. In F, $a_1, a_2, \cdots, a_{m-1}$ are the letters of a primitive constituent group. Because M is paired with itself in G_1 (§1), G contains a permutation $S=(xa)\cdots$ which transforms F into itself, G_1 into G_2 (fixing a), M of G_1 into a transitive constituent of G_2 on the letters $x, b_1, \cdots, b_{m-1}$, and the transitive constituent $a_1, a_2, \cdots, a_{m-1}$ of F into a transitive constituent $b_1, b_2, \cdots, b_{m-1}$ of F. These two transitive constituents of F are distinct because if they have one letter in common they have every letter in common and therefore $\{G_1, G_2\}$ would permute the $m+1$ letters $x, a, a_1, \cdots, a_{m-1}$ only among themselves, making G either intransitive or doubly transitive, contrary to hypothesis. Nor can the letters $b_1, b_2, \cdots, b_{m-1}$ be the letters of a transitive constituent of G_1. For if so, $\{G_1, G_2\}$ has a transitive constituent of degree m in the letters $x, b_1, \cdots, b_{m-1}$. The letters $b_1, b_2, \cdots$ belong to a transitive constituent (P) of G_1 of degree $\geqq m$.

There is an invariant subgroup H in G_1 corresponding to the identity of M. Because M is paired with itself in G_1, there is a complete set of m conjugate subgroups $H_1, H_1', \cdots$, similar to H, in G_1 which are permuted according to the transitive constituent M (§4). Let H_1 correspond to the letter a of M. The largest subgroup of G_1 in which H_1 is invariant is F; H and H_1 are the invariant subgroups of G_1 and G_2 respectively. Since H_1 is a subgroup of G_1 and is not a subgroup of H, it displaces at least one letter of M, and since it is invariant in F it displaces the $m-1$ letters $a_1, a_2, \cdots, a_{m-1}$. Since $S^{-1}H_1S=H$, H displaces the $m-1$ letters $b_1, b_2, \cdots, b_{m-1}$. Now P (of degree $\geqq m$) has an invariant intransitive subgroup in H with one transitive constituent of degree $m-1$. It is therefore an imprimitive group with systems of $m-1$ letters each. The only permutations of G_1 that permute these $m-1$ letters $b_1, b_2, \cdots, b_{m-1}$ among themselves are the permutations of F, the subgroup of G_1 that fixes a. Because M is primitive, F is a maximal subgroup of G_1 and is one of m conjugates under G_1. Then there are m systems of $m-1$ letters each in P and they are permuted according to a primitive group of degree m. That this primitive group is exactly M is evident from a consideration of the m conjugate subgroups $F, F_1, \cdots$. For F fixes a and fixes the constituent $b_1, b_2, \cdots, b_{m-1}$ of P, F_1 fixes a_1 and the constituent $b_1', b_2', \cdots, b_{m-1}'$ of P, and so on.

9. It is worth while to extend Dr. Bennett's Theorems I to VI, replacing the three given conditions by the single condition that *M is a transitive constituent of G_1 "paired with itself,"* and adding other limitations only as needed. We shall use the notation of the preceding section (§8).

Suppose G_1 has a transitive constituent Q of degree q. This constituent Q is transformed by S into a transitive constituent of G_2 which must include at least one letter new to Q.

If F permutes the letters of Q transitively, S transforms F into itself and therefore transforms this transitive constituent (A) of F on the letters of Q into a second transitive constituent (B) of F. There is no letter of Q in B. Since G_1 and G_2 cannot have transitive constituents on the same letters, G_1 has a transitive constituent of degree $>q$ in which all the letters of B occur.

If F does not permute the letters of Q transitively, the letters of at least one transitive constituent of degree l ($\geqq 1$) of a subgroup of Q found in F is replaced by S by letters new to Q.

Instead of Q, consider now M and its subgroup M_1 that fixes a. The permutation S transforms the constituent M of G_1 into a transitive constituent of G_2 on the letters $x, b_1, b_2, \cdots, b_{m-1}$. The letters $b_1, b_2, \cdots, b_{m-1}$ do not coincide with the letters $a_1, a_2, \cdots, a_{m-1}$, for $\{G_1, G_2\}$ is simply transitive. If M_1 is transitive, $\{F, S\}$ has an imprimitive constituent with the two distinct systems $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$. If M_1 is intransitive, or if M is regular, at least one transitive constituent $b_1, b_2, \cdots, b_l$ of $S^{-1}M_1S$ $(l \geqq 1)$ contains none of the letters $a_1, a_2, \cdots, a_{m-1}$.

We now impose the condition that *the order of M is $<g/n$.*

The invariant subgroup H of G_1, corresponding to the identity of M, is conjugate under G to m subgroups $H_1, H_1', \cdots$ which G_1 permutes according to the transitive constituent M. The subgroup H_1, say, is an invariant subgroup of G_2, and must displace some $j(>1)$ letters of M: $a_1, a_2, \cdots, a_j$. Thus if M is a regular group, its order is g/n.* The transform of H_1 by S is H. Let $b_1, b_2, \cdots, b_j$ be the letters by which S replaces $a_1, a_2, \cdots, a_j$. By definition H fixes all the letters of M. All the letters of the transitive constituents of G_1 to which $b_1, b_2, \cdots, b_j$ belong are displaced by H.

Suppose the letters $b_1, b_2, \cdots, b_k$ $(k \geqq j)$, new to M in the constituent of G_2 by which S replaces M, are permuted only among themselves by G_1. Clearly $k < m-1$, for if $k = m-1$, $\{G_1, G_2\}$ has a transitive constituent on the letters of M. Then $\{G_1, G_2\}$ is of degree $m+k+1<2m$. Of the letters $b_1, b_2, \cdots, b_k$, H displaces only $b_1, b_2, \cdots, b_j$ and therefore H_1 displaces only $a_1, a_2, \cdots, a_j$. The subgroup $\{H_1, H_1', \cdots, H_1^{m-1}\}$, invariant in G_1, displaces only letters of M. Being a subgroup of G of degree $<n$, it must be intransitive and therefore M, in which it is invariant, is imprimitive. If $k=j$, all the non-invariant subgroups of G_1, similar to H, are in a single conjugate set and are permuted according to the permutations of M. Each of these subgroups H_1 $H_1', \cdots, H_1^{m-1}$ is transformed into itself by H, and Theorem II is applicable to the group $\{H_1, H_1', \cdots\}$, which however is seen to fix the wrong letters. Then $k>j$.

Let it be assumed that M, *if imprimitive, is of degree* $\leqq n/2$. Another assumption that might be made is that $k=j$. This last is a weaker form of Dr. Bennett's condition upon the degree of H: that it is $n-m-1$. It follows that there is in F at least one transitive constituent (B_1) on l of the k letters $b_1, b_2, \cdots$ which is a part of a transitive constituent P of G_1 in which occur letters $c, \cdots$ new to $S^{-1}MS$. Finally put upon G_1 the condition that these l letters of B_1 are permuted transitively by H. They may now be called b_1, $b_2, \cdots, b_l$ $(1<l\leqq j)$. This transitive constituent P of G_1 is imprimitive because of its invariant intransitive subgroup in H.

If M is primitive, F is a maximal subgroup of G_1, and therefore F is the largest subgroup of G_1 by which the letters $b_1, b_2, \cdots, b_l$ are permuted only among themselves. There are m conjugate subgroups $F, F_1, \cdots$ in G_1. Hence P permutes m systems of imprimitivity, of which $b_1, b_2, \cdots b_l$ is one, according to the primitive group M.

* Rietz, loc. cit., p. 9, Theorem 7.

If M is imprimitive, F is not maximal, and the letters $b_1, b_2, \cdots, b_l$ may be transformed among themselves by a subgroup of G_1 of which F is a subgroup. Hence our transitive constituent has m' (a divisor of m) systems of imprimitivity of l letters each. These last results may be formulated as follows. The notation of this section is used.

THEOREM VI. *Let G_1 have a transitive constituent M, of order $<g/n$, paired with itself, and of degree $\leqq n/2$ if M is imprimitive. There is a transitive constituent B_1 in $S^{-1}M_1S$ on l letters new to M which is a part of a transitive constituent P of G_1 in which are letters new to M and to $S^{-1}MS$. If the letters of B_1 are permuted transitively by H, P has m systems of imprimitivity of l letters each if M is primitive, or m' ($m'>1$ and a divisor of m) systems of l letters each if M is imprimitive.*

For example, if all the transitive constituents of M_1 are primitive groups and if H_1 displaces $m-1$ or $m-2$ letters of M, then certainly the letters of B_1 are permuted transitively by H.

By putting on the restriction that $n\geqq 2m$ when M is imprimitive, and with no corresponding condition when M is primitive, the condition "if no transitive constituent of degree l occurs in G_1, where l represents the degree of any one of the transitive constituents of the subgroup of M composed of all the substitutions leaving one letter of M fixed" of Dr. Bennett's Theorems V and VI, has been avoided. In the following theorem this condition is restored in a modified form.

THEOREM VII. *Let G_1 have a transitive constituent M of order $<g/n$, paired with itself. Let those transitive constituents of M_1 whose j letters are displaced by H_1 be primitive groups. Let G_1 have no constituent of degree j. Then G_1 has an imprimitive constituent of degree $m'l$, where l is the degree of one of the transitive constituents of M_1 whose letters are displaced by H_1 and where m' divides m if M is imprimitive and is equal to m if M is primitive.*

It was seen that H_1 displaces j letters $a_1, a_2, \cdots, a_j$ of M_1 and that H displaces $b_1, b_2, \cdots, b_j$ and no other letters of $S^{-1}M_1S$. Since H_1 is an invariant subgroup of F, H_1 displaces all the letters of a transitive constituent of M_1 if it displaces one of the letters of that constituent. If G_1 does not permute $b_1, b_2, \cdots, b_j$ in one or several transitive constituents, one primitive constituent $b_1, b_2, \cdots, b_l$ of F has a transitive subgroup in H and is part with letters $c, \cdots$, new to M and to $S^{-1}MS$, of an imprimitive constituent P of degree $m'l$. Our theorem follows as before.

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